1.4 Rates of Change (Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

Average rate of change:

The average rate of change of function f over the interval (a, b) is given by this equation:

$$\boldsymbol{Average\ rate\ of\ change = \frac{f(b) - f(a)}{b - a}}$$

- Average rate of change is a measure of how much a function changes per unit, on average, over that interval.
- Average rate of change is just the slope of the straight line connecting the interval's endpoints on the graph of the function.

Average rate of example 1:

Find the average rate of change for each function over the given interval. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

$f(x) = -2x^2 + 4x - 5$ between $x = 1$ and $x = 3$

Create 2 points using the given x values.

$$x = 1; \quad f(1) = -2(1)^2 + 4(1) - 5 = -3$$

Creates the point: $(1, -3)$

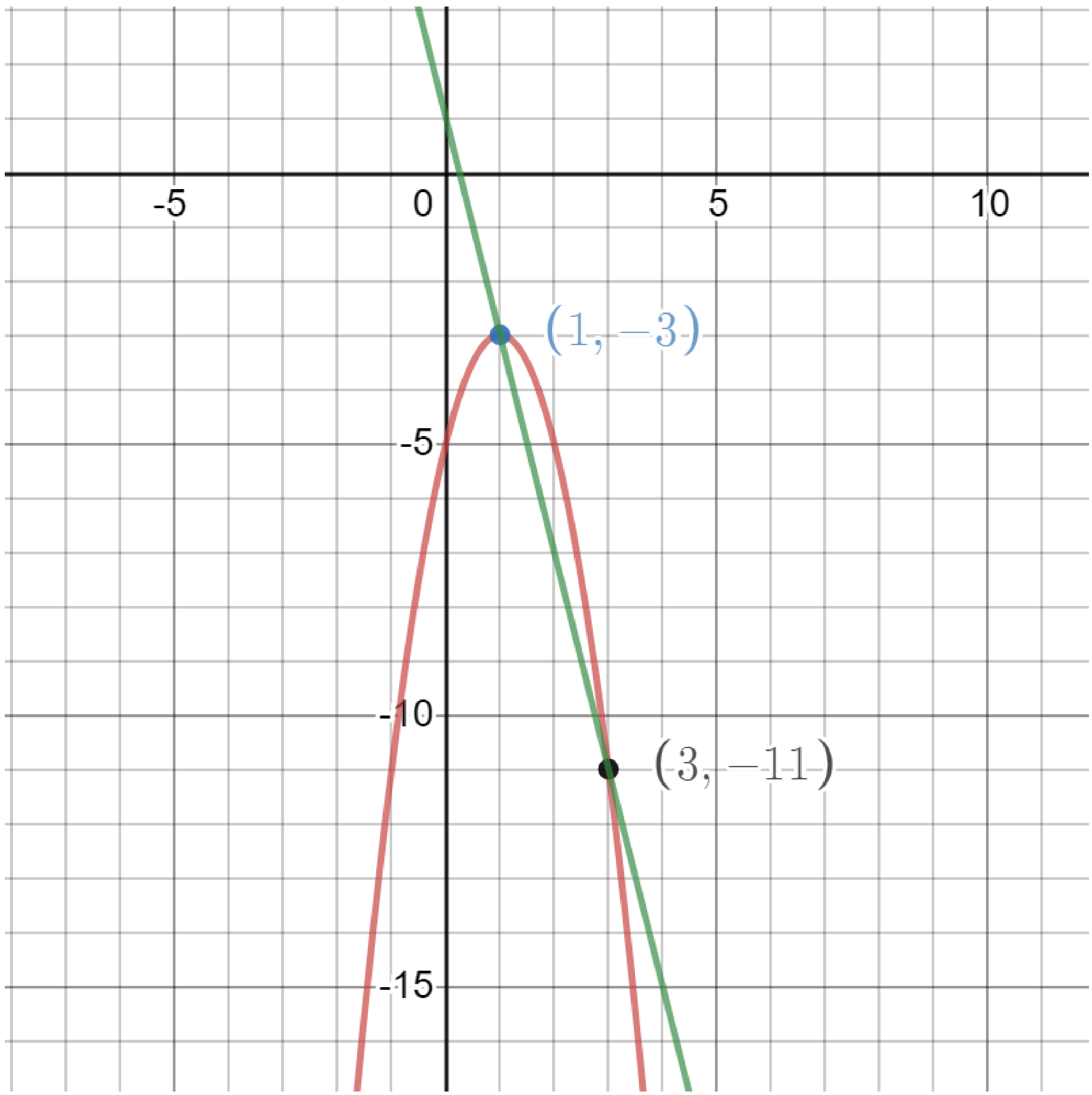
$$x = 3; \quad f(3) = -2(3)^2 + 4(3) - 5 = -11$$

Creates the point: $(3, -11)$

Average rate of change is just the slope of the line that connects the points.

$$Average\ Rate\ of\ Change = \frac{-11-(-3)}{3-1} = \frac{-8}{2} = -4$$

Answer: Average Rate of Change = -4


-5
0
5
10
-5
-10
-15
(1, −3)
(3, −11)

Average rate of change example 2:

At 10 AM a car's odometer read 10,300 miles. At noon, the car's odometer read 10,420 miles. What is the car's average rate of change measured in miles per hour?

We need to create two points.

Since we are asked to find the average rate of change in miles per hour

$x - coordinate$ of the points must be time in hours (hours are mentioned second)

$y - coordinate$ of the points must be distance in miles (miles are mentioned first)

$(time, miles)$

These are the points needed: $(10,\ 10300)\ (12,\ 10420)$

$$Average\ Rate\ of\ Change = \frac{10420 - 10300\ (miles)}{12 - 10\ \ (hours)} = \frac{120\ miles}{2\ hours} = 60\ miles\ per\ hour$$

Answer: Average Rate of Change (velocity): $60\ mph$

- Average rate of change tells us the average rate at which a function changes over an interval.

- The average rate of change only tells us an average change between two values of x. It gives no specific information in-between the two values of x.
  - That is, we have no idea how the function behaves at any specific instant in the interval between the given values of x.

- In the car example we computed the average rate of change (velocity) was $60\ miles\ per\ hour$ over the 2-hour trip. This is an average speed for the entire trip. This does not mean the car traveled at precisely $60\ mph$ for the entire trip. This is just the average speed. In fact, it likely went faster than $60\ mph$ at times and slower than $60\ mph$ at other times. The car could have been stopped for a chunk of time.
  - We need to calculate the car's **instantaneous rate of change** to know its speed (velocity) at a specific moment.

Instantaneous Rate of Change (Derivative):

- Instantaneous rate of change at $x = a$ is the average rate of change over the interval (a, a)
- Instantaneous rate of change cannot be calculated with Algebra alone because the of the equal x-coordinates.
- The denominator of the fraction in the computation would be $\frac{}{a-a}$.
- This gives a fraction with 0 in the denominator. This is undefined.

Calculus is needed to compute instantaneous rate of change.

Instantaneous rate of change of a function f at $x = a$: Is the average rate of change of the function f at $x = a$.

- The instantaneous rate is essentially the average rate of change over the interval $(a, a + h)$.
- To find the instantaneous rate of change, we take the limit as $h\ approaches\ 0$.
- The instantaneous rate of change is the average rate of change at a single point, since h is changed to zero and we get the interval $(a, a + 0) = (a, a)$

This is the formula to compute instantaneous rate of change of a function $f\ when\ x = a$.

$Instantaneous\ rate\ of\ change\ of\ a\ function\ f\ at\ x = a$

$$Instantaneous\ rate\ of\ change\ = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Instantaneous rate of change is a measure of the slope of the line connecting the points: $\big(a, f(a)\big)\ and\ \ (a + h, f(a + h))$

We will use the word DERIVATIVE very often this semester.

- The terms instantaneous rate of change and derivative have the same definition and they are interchangeable words.
- The value of a derivative of function f when $x = a$ is just the instantaneous rate of change of the function f at $x = a$

This is the formula to compute derivative of a function $f\ at\ x = a$.

$\boldsymbol{Derivative\ of\ function\ f\ at\ x = a}$

Derivative = $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

- We use the symbol $f'(a)\ to\ represent\ the\ derivative\ of\ function\ at\ x = a$
- A derivative is a measure of the slope of the line connecting the points: $\big(a, f(a)\big)\ and$ $(a + h, f(a + h))$

Instantaneous rate of change example 1:

Find the instantaneous rate of change at the given value. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

Use the instantaneous rate of change formula: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$f(x) = 3x^2 - 5x + 4; x = 2$

Create 2 points

Use $x = 2$ as the x-coordinate of the first point

Find the y-coordinate of the first point:

$$f(2) = 3(2)^2 - 5(2) + 4 = 6$$

First point: $(2,\ 6)$

Use $2 + h$ as the x-coordinate of the second point.

Find the y-coordinate of the second point:

$$f(2+h) = 3(2+h)^2 - 5(2+h) + 4$$
$$= 3(2+h)(2+h) - 5(2+h) + 4$$
$$= 3(4 + 4h + h^2) - 5(2+h) + 4$$
$$= 12 + 12h + 3h^2 - 10 - 5h + 4$$
$$= 3h^2 + 7h - 5x + 6$$

Second point: $(2+h,\ \ 3h^2 + 7h + 6)$

$$Instantaneous\ rate\ of\ change = f'(2) = \lim_{h\to 0}\frac{3h^2+7h+6-6}{h}$$

$$f'(2) = \lim_{h\to 0}\frac{3h^2+7h}{h} = \lim_{h\to 0}\frac{h(3h+7)}{h} = \lim_{h\to 0}(3h+7) = 3(0) + 7 = 7$$

Answer: Instantaneous rate of change at $x = 2\ is$ 7: $f'(2)$ = 7

Instantaneous rate of change example 2:

A pebble is dropped from a cliff, $288\ foot$ cliff. The pebble takes 3 seconds to hit the ground.
The formula: $f(t) = 288 - 32t^2$ Can be used to calculate the pebbles height of the ground in f feet $t - seconds$ after it is dropped.

a) Calculate the average rate of change (average speed) in feet per second of the pebble for the 3 seconds it takes to hit the ground.

We need to create two points.

Since we are asked to find the average rate of change (velocity) in feet per second

$x - coordinate$ of the points must be time in seconds (hours are mentioned second)

$y - coordinate$ of the points must be height in feet (feet are mentioned first)

$(seconds, feet)$

These are the points needed:

$(0,\ 288)$ (at 0 seconds the pebble is 288 feet high)

$(3,\ 0)$ (at 3 seconds the pebble is on the ground.

$Average\ Rate\ of\ Change = \frac{0-288\ (feet)}{3-0\ \ (seconds)} = \frac{-288\ feet}{3\ seconds} =$
$-96\ feet\ per\ second$

Answer: Average Rate of Change (average speed):
$-96\ feet\ per\ second$ (answer is negative since the pebble is falling)

b) Calculate the instantaneous rate of change in feet per second (velocity) of the pebble at t = 3 seconds.

Create 2 points
Use $x = 3$ as the x-coordinate of the first point

Find the y-coordinate of the first point:

$$f(3) = 288 - 32(3)^2 = 0$$

First point: $(3,\ 0)$

Use $3 + h$ as the x-coordinate of the second point.

Find the y-coordinate of the second point:

$$f(3 + h) = 288 - 32(3 + h)^2$$
$$= 288 - 32(3 + h)(3 + h)$$
$$= 288 - 32(9 + 6h + h^2)$$
$$= 288 - 288 - 192h - 32h^2$$
$$= -192h - 32h^2$$

Second point: $(3 + h,\ -192h - 32h^2)$

$$Instantaneous\ rate\ of\ change\ (velocity) = f'(3) = \lim_{h \to 0} \frac{192h - 32h^2 - 0\ \ (feet)}{h\ (seconds)}$$

$$f'(3) = \lim_{h \to 0} \frac{h(-192 - 32h)(feet)}{h\ \ (seconds)} = \lim_{h \to 0}(-192 - 32h) = -192 - 32(0) = -192\ feet\ per\ second$$

Answer: The pebble's velocity (instantaneous rate of change) is -192 *feet per second* when it hits the ground. (The negative sign indicates the pebble is falling.)

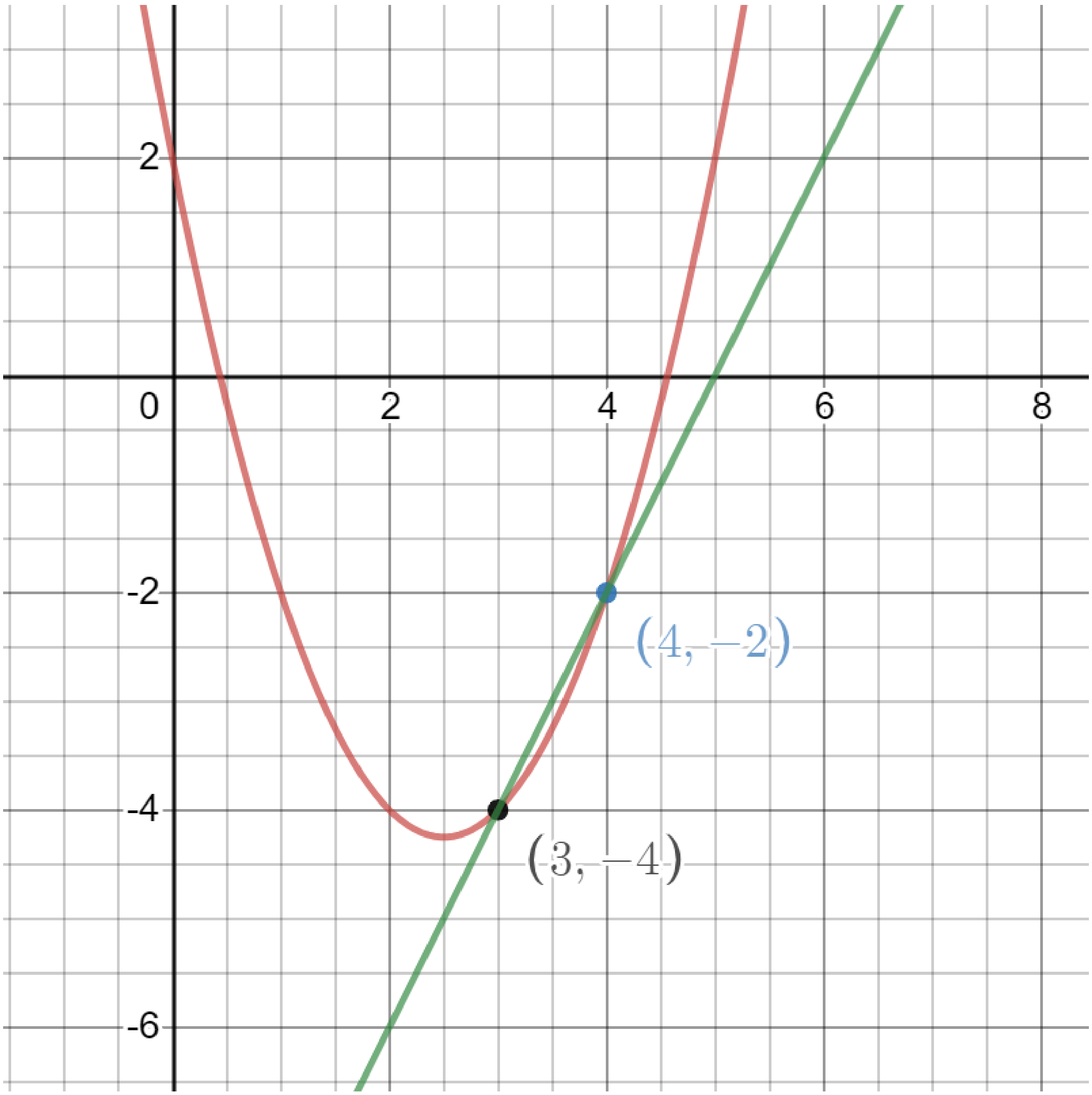
(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

#1-8: Find the average rate of change for each function over the given interval. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

1) $f(x) = x^2 - 5x$ between $x = 3$ and $x = 4$

2) $f(x) = x^2 - 5x + 2$ between $x = 3$ and $x = 4$

Answer: Average rate of change = 2



2
0
2
4
6
8
-2
-4
-6
(4, −2)
(3, −4)

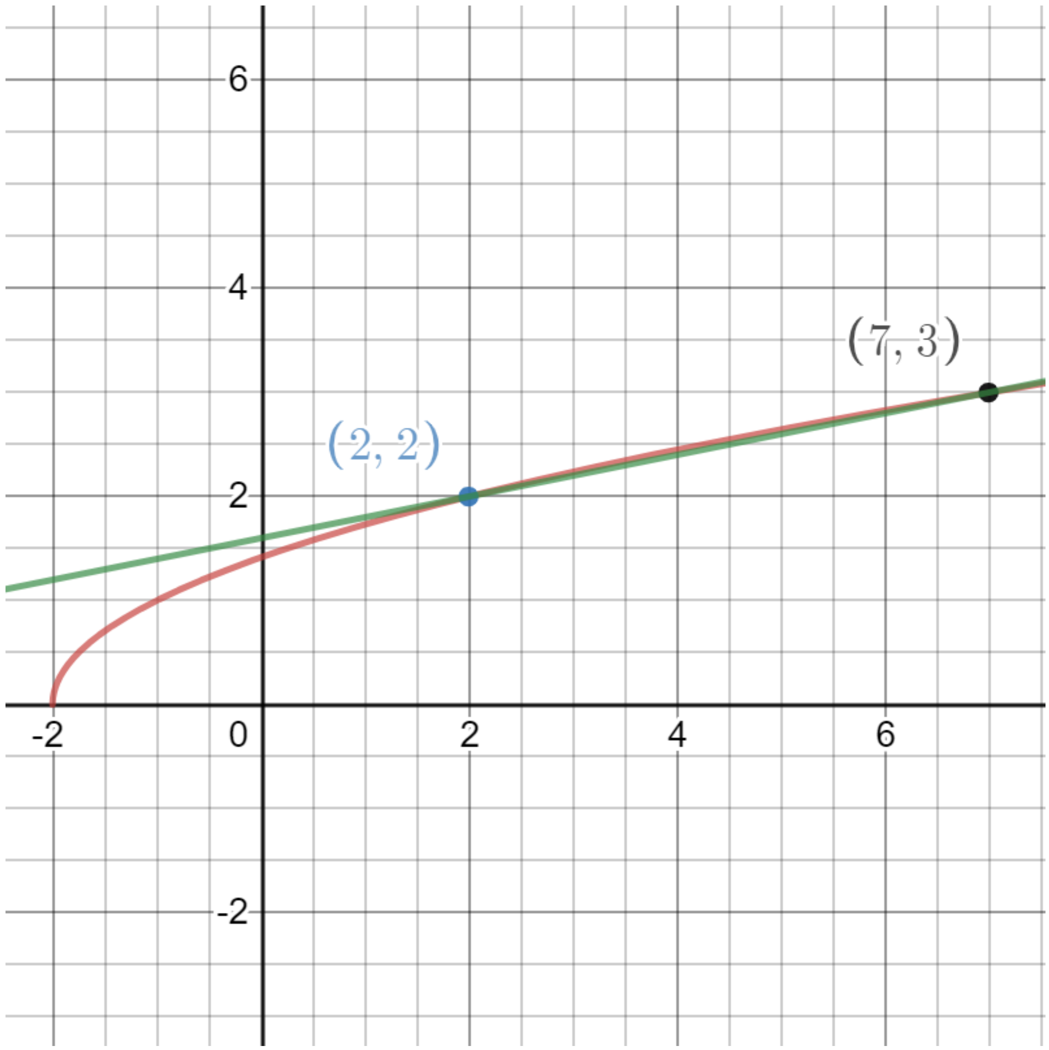
(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

#1-8: Find the average rate of change for each function over the given interval. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

3) f(x) = $\sqrt{x-5}$ between x = 9 and x = 14

4) f(x) = $\sqrt{x+2}$ between x = 2 and x = 7

Answer: Average rate of change = $\frac{1}{5}$



6
4
2
-2
-2
0
2
4
6
(2, 2)
(7, 3)

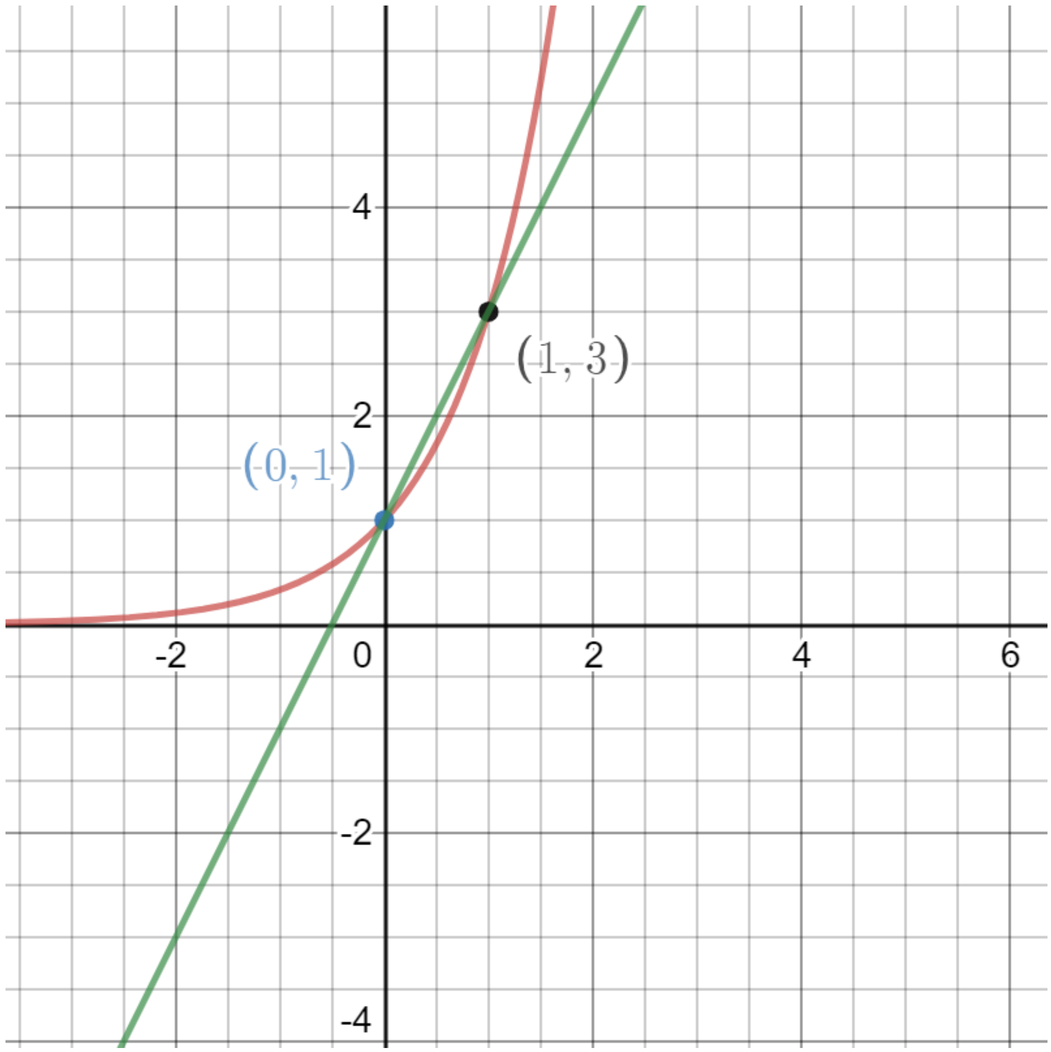
(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

#1-8: Find the average rate of change for each function over the given interval. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

5) $s(t) = 2^t$ between $t = 0$ and $t = 2$

6) $s(t) = 3^t$ between $t = 0$ and $t = 1$

Answer: Average rate of change = 2



4
2
-2
-4
-2
0
2
4
6
(0, 1)
(1, 3)

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

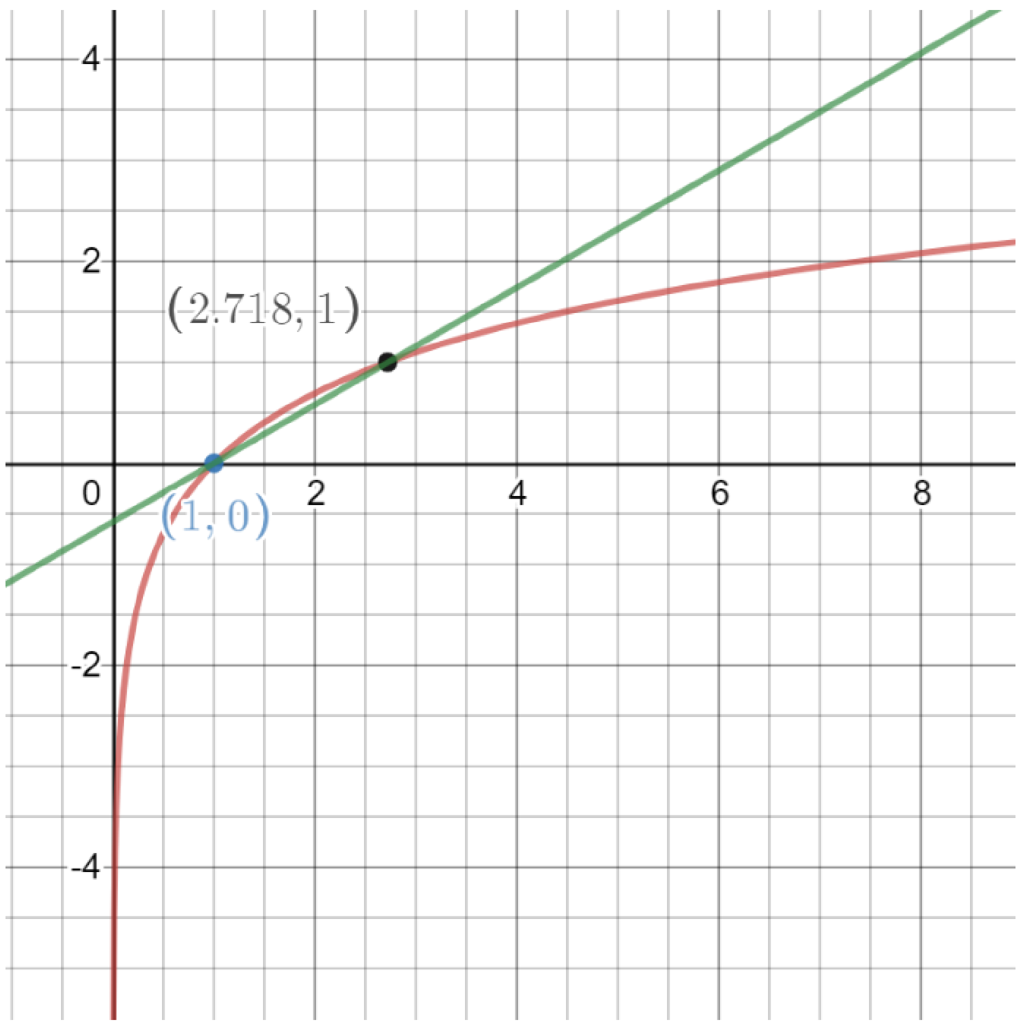
#1-8: Find the average rate of change for each function over the given interval. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

7) c(t) = ln(t) between t = 1 and t = e

8) c(t) = ln(t) between t = 1 and t = e^2

Answer for problem 7. (I will solve 7 in the video instead of 8)

Average rate of change = $\frac{1}{e-1} \approx 0.58$



4
2
(2.718, 1)
0
2
4
6
8
(1, 0)
-2
-4

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

9) A climber is on a hike. After 2 hours he is at an altitude of 400 feet. After 6 hours, he is at an altitude of 700 feet. What is the average rate of change in the climber's height?

10) A scuba diver is 30 feet below the surface of the water 10 seconds after he entered the water and 50 feet below the surface after 40 seconds. What is the scuba divers' average rate of change in the diver's depth measured in feet per second? (write answer as a reduced fraction)

Answer: The scuba divers' average rate of change is $\frac{2}{3}\ feet\ per\ second$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

11) A rocket is 1 mile above the earth in 30 seconds and 5 miles above the earth in 2.5 minutes. What is the rockets average rate of change in the height of the rocket in miles per second?

12) A teacher weighed 160 lbs in 1996 and weighs 210 lbs in 2013. What was the average rate of change in weight? (round to 2 decimals)

Answer: The average rate of change is 2.94 pounds per year.

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

13) This problem has been deleted.

14) Michael started a savings account with $300. After 4 weeks, he had $350 dollars, and after 9 weeks, he had $400. What is the average rate of change of money in his savings account per week?

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

15) A plane left Chicago at 8:00 A.M. At 1: P.M., the plane landed in Los Angeles, which is 1500 miles away. What was planes average rate of change miles per hour?

16) After 30 baseball games, a baseball player had 25 hits. If after 100 games he had 80 hits, what are his average hits per baseball game?

Answer: The player averaged 4 hits per game (nice!!)

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

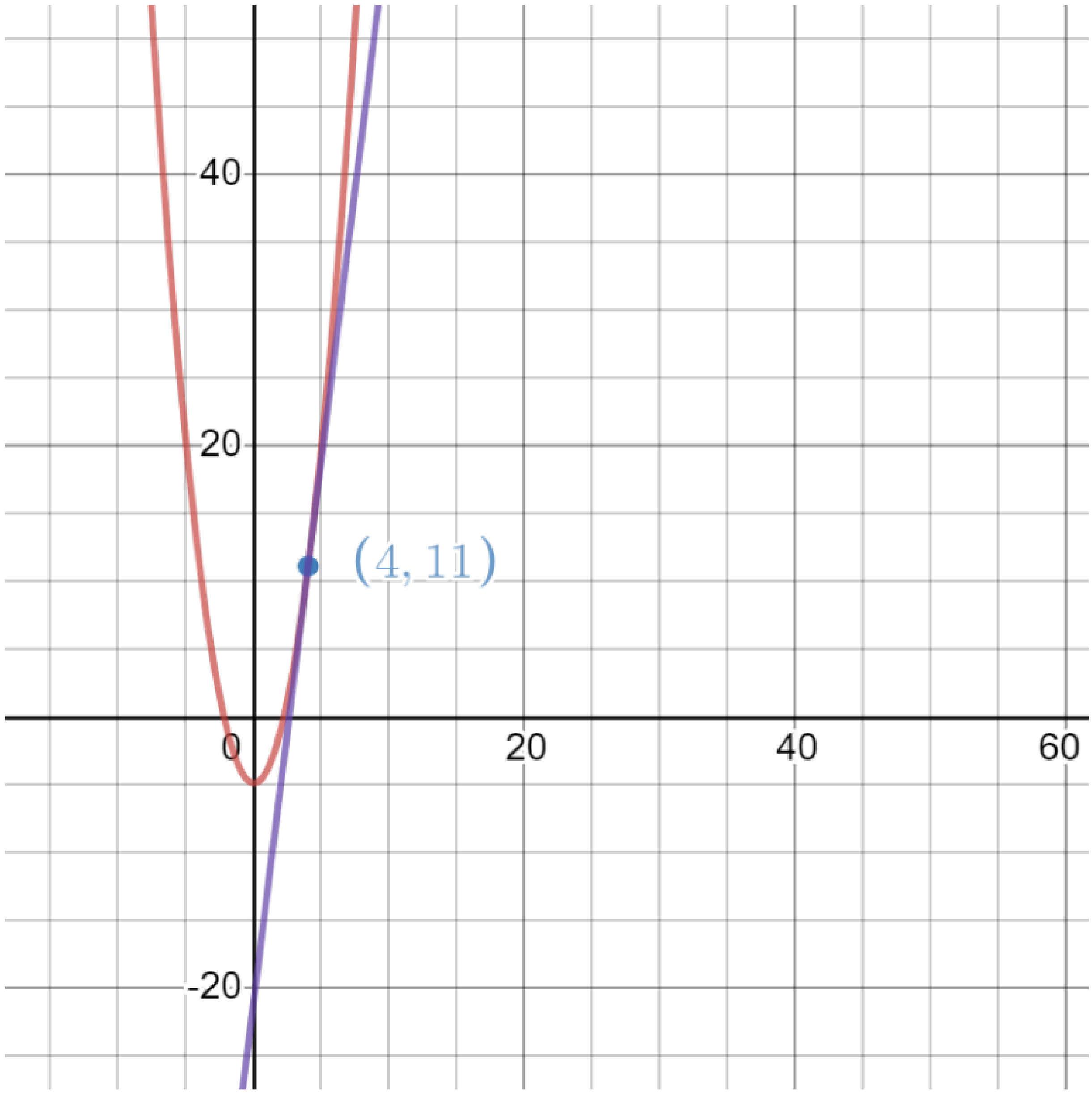
#17 – 24: Find the instantaneous rate of change at the given value. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

Use the instantaneous rate of change formula: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

17) f(x) = $x^2 - 3$ at x = 2

18) f(x) = $x^2 - 5$ at x = 4

Answer: $f'(4) = 8$ $\;(equation\ of\ tangent\ line\ \ y = 8x - 21$



40
20
-20
0
20
40
60
(4, 11)

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

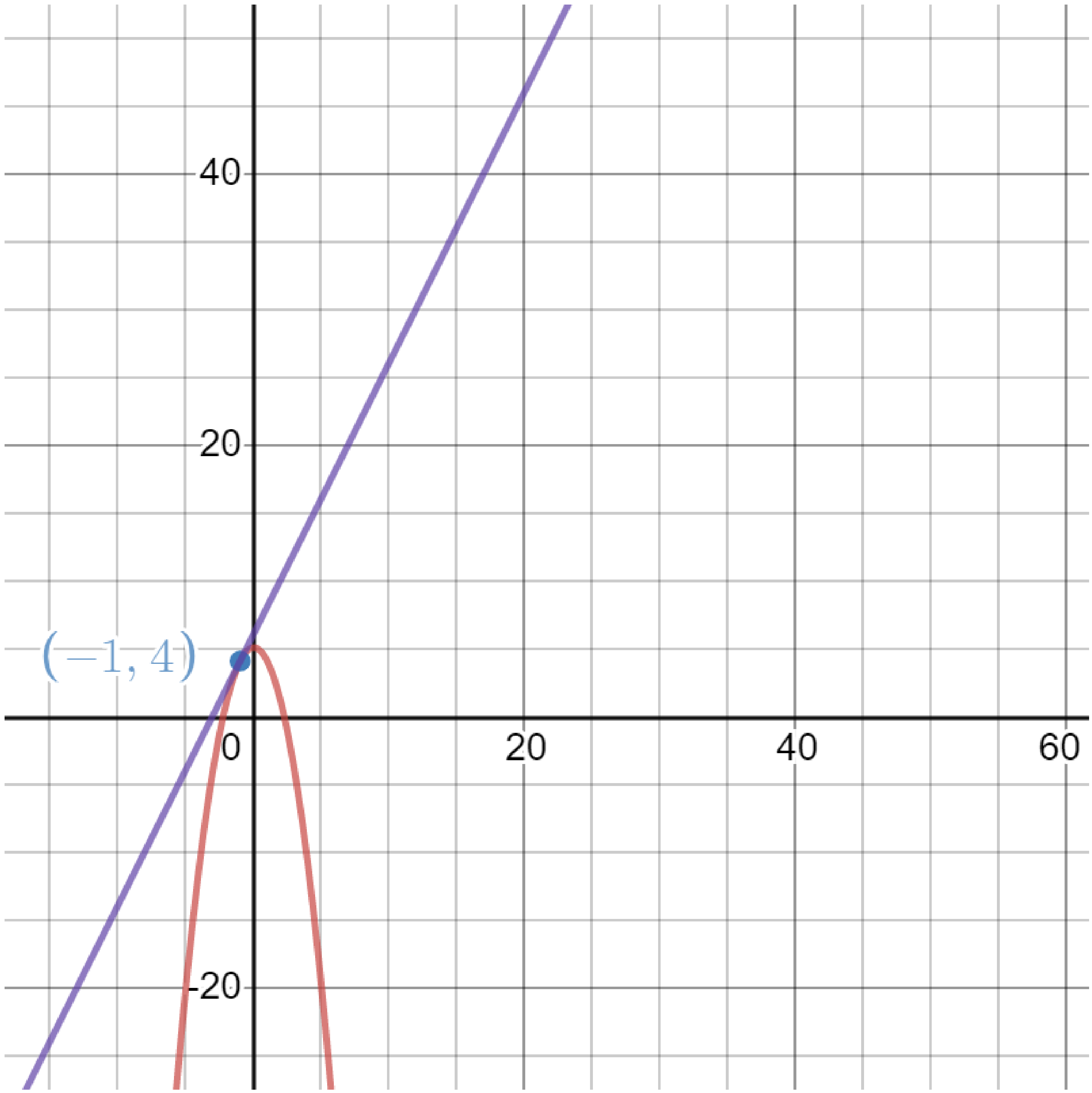
#17 – 24: Find the instantaneous rate of change at the given value. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

Use the instantaneous rate of change formula: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

19) $f(x) = 3 - x^2$ at $x = 1$

20) $f(x) = 5 - x^2$ at $x = -1$

$answer\text{: } f'(-1) = 2 \;\; (equation\ of\ tangent\ line\ \ y = \ 2x + 6)$



40
20
(−1, 4)
0
20
40
60
-20

(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

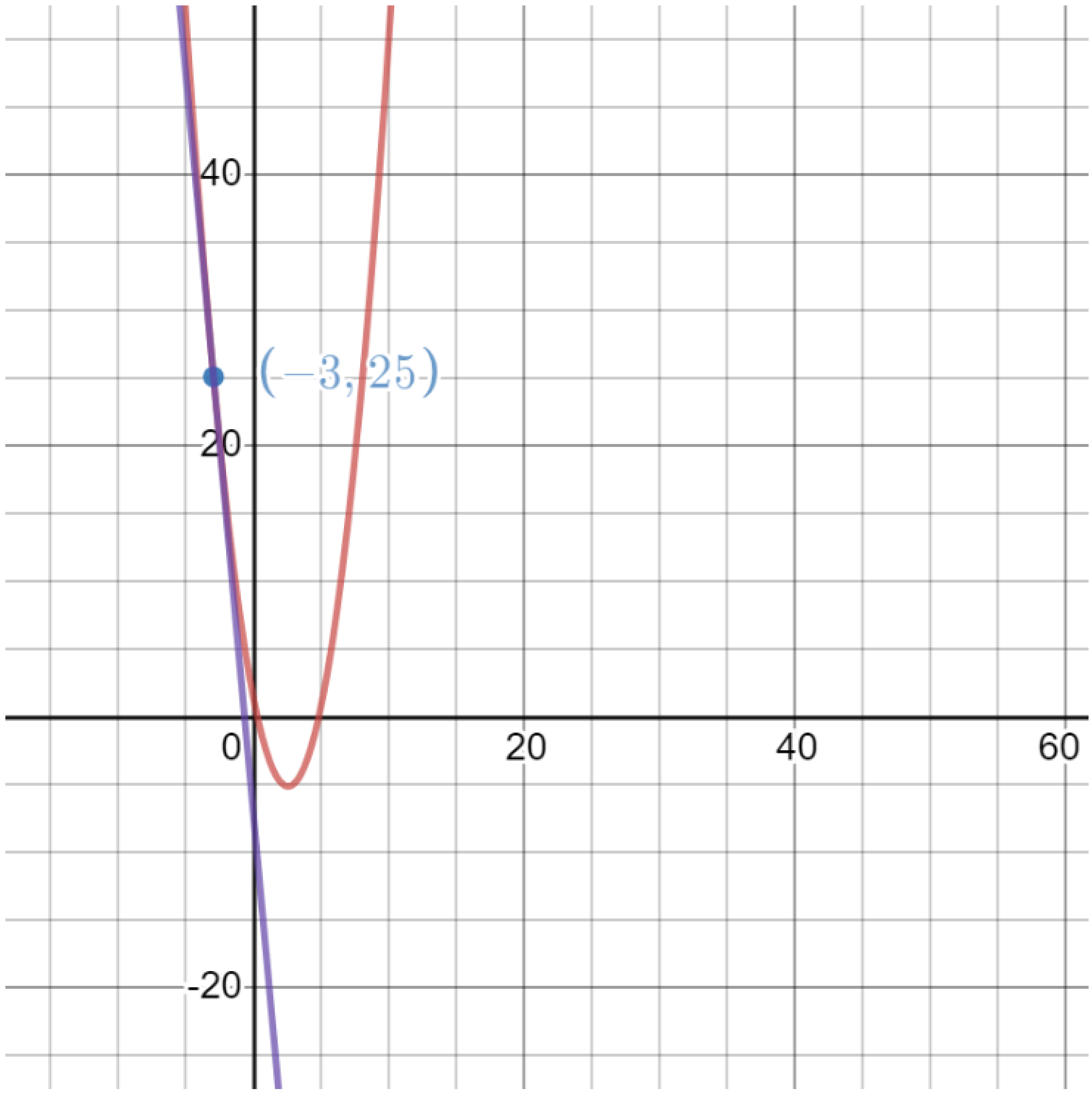
#17 – 24: Find the instantaneous rate of change at the given value. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)

Use the instantaneous rate of change formula: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

21) $g(t) = t^2 + 2t - 3$ at $t = -2$

22) $g(t) = t^2 - 5t + 1$ at $t = -3$

Answer: $g'(-3) = -11$ (equation of tangent line $y = -11x - 8$)



40
20
0
-20
20
40
60
(−3, 25)

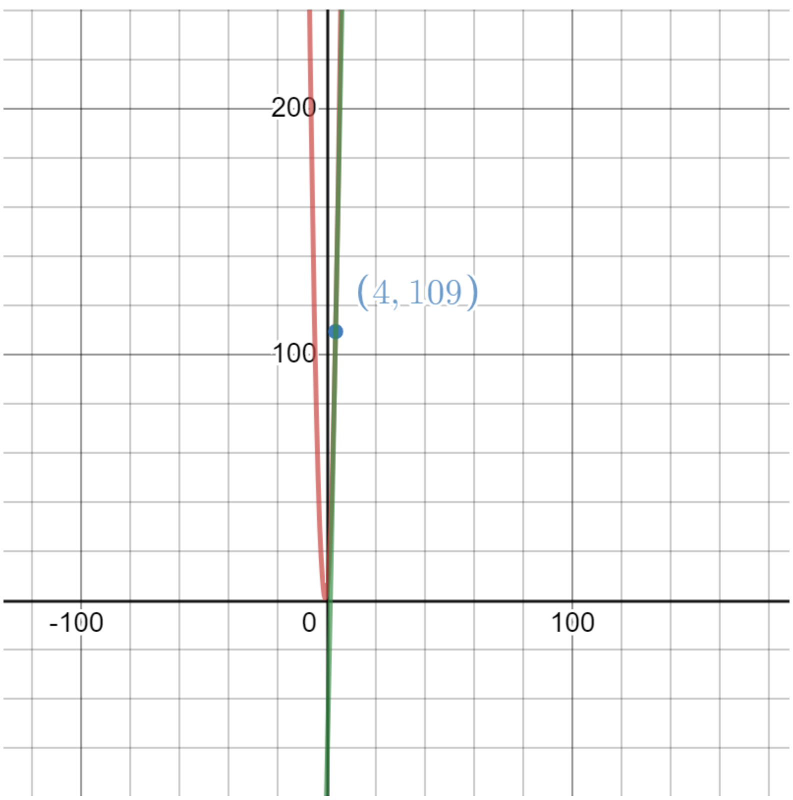
(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

#17 – 24: Find the instantaneous rate of change at the given value. Sketch a graph to model your answer. (You may use your calculator obtain the graph, be sure to label the necessary points.)
Use the instantaneous rate of change formula: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

23) $h(t) = 5t^2 - 2t + 3$ at $t = 0$

24) $h(t) = 6t^2 - 3t + 1$ at $t = 4$

Answer: $h'(t) = 51$ (equation of tangent line $y = 51x - 95$)



(Minimum Homework: 1, 3, 5, 7, 9, 11, 15, 19, 21, 25)

25) A toy rocket is launched straight up so that its height s, in meters, at time t, in seconds, is given by $s(t)=-2t^2+30t+5$. Calculate the instantaneous rate of change (velocity) of the rocket at $t=3$.

26) If a baseball is projected upward from ground level with an initial velocity of 64 feet per second, then its height is a function of time, given by $s(t) = -16t^2 + 64t$. Calculate the instantaneous rate of change (velocity) of the ball at t = 2 seconds.

Answer: The ball's velocity (instantaneous rate of change) is
$0\ feet\ per\ second$
(Velocity of 0 ft / sec indicate the object has reached its maximum height)

27) A pebble is dropped from a cliff, 50 m high. After *t* sec, the pebble is *s* meters above the ground, where $s(t)=50-2t^2$. Calculate the instantaneous rate of change (velocity) of the pebble at t = 2 seconds.

28) A cannon ball is dropped from a building. Suppose that the height of the cannon ball (in meters) after t seconds is given by the quadratic function: $f(t) = -4.4t^2 + 50$. Calculate the instantaneous rate of change (velocity) of the ball at t = 1 seconds.

29) The profit from sale of x car seats is given by the formula: $P(x)= 45x - 0.0025x^2 - 5000$

a) Find the profit from selling 800 car seats.

b) Find the instantaneous rate of change at a production level of 800 car seats. (The instantaneous rate of change of a profit function is often called the marginal profit.)

30) The profit from sale of x cell phones is given by the formula: $P(x)= 450x - 0.055x^2 - 300000$

a) Find the profit from selling 1000 cell phones.

b) Find the instantaneous rate of change at a production level of 1000 cell phones. (The instantaneous rate of change of a profit function is often called the marginal profit.)

31) The cost of manufacturing x chairs is given by the function: $C(x) = x^2 + 40x + 800$

a) Find the cost of producing 30 chairs.

b) Find the instantaneous rate of change when 30 chairs are produced. (The instantaneous rate of change of a cost function is often called the marginal cost.)

32) The cost of manufacturing x books is given by the function: $C(x) = x^2 + 30x + 50$

a) Find the cost of producing 50 books.

b) Find the instantaneous rate of change when 50 books are produced. (The instantaneous rate of change of a cost function is often called the marginal cost.)